IITJEE-2009

TS7 Paper 2

Maths Solutions

Math-Solutions Test Series – 7/Paper – II/ JEE – 2009

1. (C) For $n \in I$

$$\lim_{x\to n^+} f(x) = \lim_{x\to n^+} [x] cos\left(\frac{2x-1}{2}\right)\pi = x.cos\left(\frac{2x-1}{2}\right).\pi = 0$$

$$\lim_{x\to n^-} f(x) = \lim_{x\to n^-} [x] cos\left(\frac{2x-1}{2}\right)\pi = (x-1) cos\left[\frac{2x-1}{2}\right]\pi = 0$$

Hence f is continuous for $x = n \in I$. Since the function $[x]$ and $cos\left(\frac{2x-1}{2}\right)\pi$ are continues

$\forall\ x \in R - I$, So $f(x)$ is continuous every where.

2. (A) Vertex of $f(x)$ will be at $x = \frac{a}{2}$

Case – I If a $a < 0$, then maximum value of $f(x)$ is at $x = 0$

$\therefore f(0) = -5 \Rightarrow a^2 + 2a - 3 = 0$

$\Rightarrow a = -3$ or $1 \Rightarrow a = -3$

Case – II If $a > 2$, then maximum value of $f(x)$ is at $x = 1$

$\therefore\ f(1) = -5 \Rightarrow a^2 + a - 2 = 0$

$\Rightarrow a = -2$ or 1 which is not possible.

Case – III If $0 < a < 2$, then maximum value is at $x = \frac{a}{2}$

$$\Rightarrow\ f\left(\frac{a}{2}\right) = -5\ \Rightarrow 3a^2 + 8a - 12 = 0$$

$$\Rightarrow\ a = \frac{-4 \pm 2\sqrt{13}}{3} \Rightarrow a = \frac{-4 + 2\sqrt{13}}{3}$$ is possible

3. (B) In the neighborhood of $x = -\frac{\pi}{6}$, we have $|x| = -x$ and $|tan\,x| = -tan\,x$

$\therefore\ f(x) = (-x)^{-tan\,x}$

$\Rightarrow\ f(x) = e^{-tan\,x.log(-x)}$

$$\Rightarrow\ f'(x) = (-x)^{-tan\,x}\left\{-sec^2 x.log(-x) - \frac{tan\,x}{x}\right\}$$

$$\Rightarrow\ f'\left(-\frac{\pi}{6}\right) = \left(\frac{\pi}{6}\right)^{1/\sqrt{3}}\left\{-\frac{4}{3} log\frac{\pi}{6} - \frac{2\sqrt{3}}{\pi}\right\}$$

$$\Rightarrow\ f'\left(-\frac{\pi}{6}\right) = \left(\frac{\pi}{6}\right)^{1/\sqrt{3}}\left\{\frac{4}{3} log\frac{6}{\pi} - \frac{2\sqrt{3}}{\pi}\right\}$$

4. (B) Given that $sin\,x + cosec\ x + tan\,y + cot\,y = 4$

$\Rightarrow \quad x = \frac{\pi}{2}$ and $y = \frac{\pi}{4}$

$\Rightarrow \quad \tan y = 1$

$\Rightarrow \quad \frac{2\tan y/2}{1-\tan^2 y/2} = 1 \;\Rightarrow\; \tan^2\frac{y}{2} + 2\tan\frac{y}{2} - 1 = 0$

5. (B) $0 \le \{x\} < 1 \Rightarrow 0 \le 2\{x\} < 2$

$\therefore$ Maximum value of $\left[2\{x\}\right] = 1$

6. (B) Atleast one green toy can be selected out of 6 different toys in

$${}^6C_1 + {}^6C_2 + \ldots\ldots\ldots\ldots + {}^6C_6 = 63 \text{ ways or } 2^6 - 1 = 63 \text{ ways}$$

After selecting one or more green toys we can select atleast one toy out of 5 different blue toys in

$${}^5C_1 + {}^5C_1 + \ldots\ldots\ldots\ldots + {}^5C_5 = 31 \text{ ways or } 2^5 - 1 = 31 \text{ ways}$$

After selecting atleast one green toy and atleast one blue toy.
Selection of red (no restriction) can be made in

$${}^4C_0 + {}^4C_1 + \ldots\ldots\ldots\ldots + {}^4C_4 = 16 \text{ ways or } 2^4 = 16 \text{ ways}$$

Total number of selections $= 63 \times 31 \times 16 = 31248$.

7. (D) Equation of circle will be $(x+r)^2 + y^2 = r^2$, on differentiating both sides, we get

$$2(x+r) + 2y.\frac{dy}{dx} = 0$$

$\therefore\; r = -yy_1 - x$, substituting in $(x+r)^2 + y^2 = r^2$

$\Rightarrow \quad y^2y_1^2 + y^2 = (x + yy_1)^2$

$\Rightarrow \quad y^2 = x^2 + 2xyy_1$

$\Rightarrow \quad 2xy\frac{dy}{dx} = y^2 - x^2$.

8. (B) Any line passing through (2, 2) will be of the form

$$\frac{y-2}{\sin\theta} = \frac{x-2}{\cos\theta} = r$$

When this line cuts the circle

$$x^2 + y^2 = 2$$

$\therefore\; (r\cos\theta + 2)^2 + (r\sin\theta + 2)^2 = 2$

$\Rightarrow \quad r^2 + 4(\sin\theta + \cos\theta)r + 6 = 0$

$\frac{PB}{PA} = \frac{r_2}{r_1}$, now if $r_1 = \alpha, r_2 = 3\alpha$

Then $4\alpha = -4(\sin\theta + \cos\theta), 3\alpha^2 = 6$

$\Rightarrow \quad \sin 2\theta = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$.

So, required chord will be $y-2=1(x-2) \Rightarrow y=x$.

Alternate solution :

$PA\,.\,PB = PT^2 = 2^2 + 2^2 - 2 = 6$ ………………. (I)

Given $\dfrac{PB}{PA} = 3$ ………………. (II)

From equations (I) and (II), we have

$PA = \sqrt{2}, PB = 3\sqrt{2}$

$\Rightarrow \quad AB = 2\sqrt{2}$. Now diameter of the circle is $2\sqrt{2}$ (as radius is $\sqrt{2}$)

Hence line passing through the centre

$\Rightarrow \quad y = x$

9. (C) Since, $0 < \dfrac{3}{x^2+1} \le 3$

$$\Rightarrow \begin{cases} 0 < \dfrac{3}{x^2+1} < 1 & \text{when} \quad x > \sqrt{2} \\ 1 \le \dfrac{3}{x^2+1} < 2 & \text{when} \quad \dfrac{1}{\sqrt{2}} < x \le \sqrt{2} \\ 2 \le \dfrac{3}{x^2+1} < 3 & \text{when} \quad 0 < x \le \dfrac{1}{2} \end{cases}$$

$$\therefore\ I = \int_0^{1/\sqrt{2}} 2\,dx + \int_{1/\sqrt{2}}^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\infty} 0\,dx$$

$$= \sqrt{2} + \sqrt{2} - \frac{1}{\sqrt{2}} = 2\sqrt{2} - \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}.$$

10. (C) **Cancelled**

11.(A) N.A.

12. (C) **N.A.**

13. (A) $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar

$$\therefore\ (\vec{b} - \vec{a})\,.\,\left[(\vec{c} - \vec{a}) \times (\vec{d} - \vec{a})\right] = 0$$

$\Rightarrow$ After simplifying we get required result.

14. (B) $$I_n = \int_0^a e^{-x} x^n\,dx = x^n\left(-e^{-x}\right)\Big|_0^a - \int_0^a n\,x^{n-1}\left(-e^{-x}\right)dx$$

$$= -e^{-a}\,.\,a^n + nI_{n-1}$$

$\Rightarrow I_n - nI_{n-1} = -e^{-a}\,.\,a^n$ ……………… (I)

15. (A) $$I_0 = \int_0^a e^{-x}dx = -e^{-x}\Big|_0^a = -e^{-a} + 1$$

On putting $n=1$ in (I) of 14 we get,

$I_1 = I_0 - e^{-a} . a^1$

$= -e^{-a} + 1 - e^{-a} . a$

$(1 - I_1)e^{+a} = 1 + a$

Again putting n = 2

$I_2 = 2I_1 - e^{-a} . a^2$

$= 2\left[-e^{-a} + 1 - a\, e^{-a}\right] - e^{-a} a^2$

$\Rightarrow \left(1 - \frac{I_n}{2!}\right) e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + ... + \frac{a^n}{n!}$

16. (B) Since as $a \to \infty$, RHS of

$$1 - \frac{I_n}{n!} = \lim_{n\to\infty} \frac{1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + ... + \frac{a^n}{n!}}{1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + ... + \frac{a^n}{n!} + \frac{a^{n+1}}{(n+1)!}} = 0$$

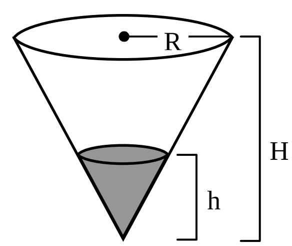
$\Rightarrow \frac{I_n}{n!} \to 1$ as $a \to \infty \Rightarrow I_n \to n!$ as $a \to \infty$

17, 18, 19 Let θ be the semi vertical angle of the cone so that $\tau an\theta = \frac{R}{H}$

(B, A, C) Let the radius and height of water cone at time *t* be *r* and *h* respectively. So, $tan\,\theta = \frac{r}{h}$

If *V* is the volume of water and *S* is the surface of the cone is contact with air at time *t*, then

$V = \frac{1}{3}\pi r^2\, h = \frac{1}{3}\pi r^3 \cot\theta$ and $S = \pi r^2$



We are given that $\frac{dV}{dt} \propto S$

$\Rightarrow \frac{dV}{dt} = -kS$ (V is decreasing)

$\Rightarrow \frac{1}{3}\pi\left(3r^2\right)\cot\theta \frac{dr}{dt} = -k\left(\pi r^2\right) \Rightarrow \frac{dr}{dt} = -k\tan\theta$

Integrating, we get

$r = -k(tan\,\theta)t + C$

When $t = 0, r = k \therefore C = R$

Thus $r = (-k\,tan\,\theta)t + R$

When cone is empty, $r = 0$. If T is the time taken $0 + (-k\,tan\,\theta)T + R$

$\Rightarrow T = \frac{R}{k\,tan\,\theta} = \frac{R}{k\left(\frac{R}{H}\right)} = \frac{H}{k}$

Hence, the cone will be empty in time $\frac{H}{k}$.

$$r(1) = -R\tan\theta + R = -k\frac{R}{H} + R = R\left(1 - \frac{k}{H}\right)$$

$$\sum_{i=1}^{10} r(i) = 10R - \frac{R}{H}k\sum_{i=1}^{10} i = 10R - \frac{Rk}{H}55$$

20. **CANCELLED**

21. **(A)** Sum of the series in β = $\beta^{15}\left(1+\beta+\beta^2+..........+\beta^{36}\right)$

$$= \beta^{15}\cdot\frac{\beta^{36}-1}{\beta-1} = \beta^{15}\cdot\frac{\beta-1}{\beta-1} \qquad \left(\because \beta^5 = 1 \Rightarrow \beta^{35} = 1\right)$$

$= 1$

(B) D

(C) By plotting the graph we easily get maximum value = 6

(D) $$f(3) = [3] + \sum_{r=1}^{2008} \frac{3+r-[3+r]}{2008}$$

$$= 3 + \sum_{r=1}^{2008} \frac{(3+r)-(3+r)}{2008} \qquad \left(\because [3+r] = 3+r\right).$$

$= 3 + 0 = 3$

22. **(A)** Integral $= \displaystyle\int_{-2008}^{2008} \frac{1+x^9-\sqrt{1+x^{18}}}{2x^9}dx$

$$= \int_{-2008}^{2008} \frac{dx}{2x^9} + \int_{-2008}^{2008} \frac{dx}{2} - \int_{-2008}^{2008} \frac{\sqrt{1+x^{18}}}{2x^9}dx$$

$$= 0 + \int_{0}^{2008} dx - 0 = 2008$$

(B) $x = r\cos\theta, y = r\sin\theta$

We get $r^2 + r^2\sin\theta\cos\theta = 60$

$$r^2 = \frac{60}{1+\frac{1}{2}\sin(2\theta)}$$

Now minimum $r^2 = \dfrac{60}{1+\frac{1}{2}} = 40$

$\Rightarrow$ Minimum $r = \sqrt{40}$

(C) $\sin\left(\cos^{-1}x\right) = \sin\left(\frac{\pi}{2} - \sin^{-1}x\right) = \cos\left(\sin^{-1}x\right)$ hence integral.

$= \int_{-1}^{1} \left(tan^{-1}\theta + cot^{-1}\theta \right) d\theta$ where $\theta = cos\left(sin^{-1} x \right)$

$= \int_{-1}^{1} \frac{\pi}{2} d\theta = \pi$

(D) We know that $\frac{A}{2}, \frac{B}{2} \in \left(0, \frac{\pi}{2} \right)$

Case (I)

Let $\frac{A}{2} > \frac{B}{2}$, then $sin\frac{A}{2} > sin\frac{B}{2}, cos\frac{A}{2} < cos\frac{B}{2}$

$a_1 > a_2, a_3 < a_4$ or $\frac{1}{a_3} > \frac{1}{a_4} \Rightarrow \frac{a_1}{a_3} > \frac{a_2}{a_4}$

$\Rightarrow \left(\frac{a_1}{a_2} \right)^{2007} > \left(\frac{a_3}{a_4} \right)^{2006}$

$\Rightarrow$ Given equality is not possible

Similarly if $\frac{A}{2} < \frac{B}{2}$ we arrive at a contradiction

Thus $\frac{A}{2} = \frac{B}{2} \Rightarrow A = B$ and triangle is isosceles.

$\Rightarrow AC = BC = 1$