# IITJEE-2009

# TS7 Paper 1

# Maths Solutions

# Math-Solutions Test Series – 7/Paper – I/ JEE – 2009

**1. (C)** $8f(x)+6f\left(\frac{1}{x}\right)=x+5$ ……………… (I)

Replace $x \to \frac{1}{x}$

$8f\left(\frac{1}{x}\right)+6f(x)=\frac{1}{x}+5$ ……………….. (II)

From (I) and (II) we get

$f(x)=\frac{1}{28}\left(8x-\frac{6}{x}+10\right)$

$y=x^2 f(x)=\frac{1}{28}\left(8x^2-6x+10x^2\right)$

$\left.\frac{dy}{dx}\right|_{-1}=\frac{1}{28}(24-20-6)=-\frac{1}{14}$

**2. (B)** $[\sin x]=-1,0,1$

If $[\sin x]=-1$ then $|\cos x|=2$ not possible

If $[\sin x]=0$ then $|\cos x|=x=\pi,2\pi,3\pi,4\pi$ are solutions

If $[\sin x]=1$ then $|\cos x|=0 \Rightarrow x=\frac{5\pi}{2}$

$\therefore$ Number of solution in $[\pi,4\pi]$ is 5

**3. (A)** Let vector $\vec{r}$ be coplanar to $\vec{a}$ and $\vec{b}$.

$\therefore \quad \vec{r}=\vec{a}+t\vec{b}$

$\Rightarrow \quad \vec{r}=\left(\hat{i}+2\hat{j}+\hat{k}\right)+t\left(\hat{i}-\hat{j}+\hat{k}\right)$

$=\hat{i}\,(1+t)+\hat{j}\,(2-t)+\hat{k}\,(1+t)$

The projection of $\vec{r}$ on $\vec{c}=\frac{1}{3}$ (Given)

$\Rightarrow \quad \frac{\vec{r}\cdot\vec{c}}{|\vec{c}\,|}=\frac{1}{\sqrt{3}}$

$\Rightarrow \quad \frac{|1\cdot(1+t)+1\cdot(2-t)-1\cdot(1+t)|}{\sqrt{3}}=\frac{1}{\sqrt{3}}$

$\Rightarrow \quad (2-1)=\pm 1$

$\Rightarrow \quad t=1 \text{ or } 3$

when, $t=1$ we have $\vec{r}=2\hat{i}+\hat{j}+2\hat{k}$

when, $t=3$ we have $\vec{r}=4\hat{i}-\hat{j}+4\hat{k}$

**4. (B)**

4. $\sin x + \csc x + \tan y + \cot y = 4$

$$\left(\sin x + \frac{1}{\sin x}\right) + \left(\tan y + \frac{1}{\tan y}\right) = 4 \quad \text{- - - - (i)}$$

$\forall\ x \in (0, \pi/2)$, $\sin x > 0$, $\tan y > 0$

$\Rightarrow \sin x + \frac{1}{\sin x} \geq 2$ $\left[\because \text{Sum of a positive number and its reciprocal} \geq 2\right]$

and $\tan y + \frac{1}{\tan y} \geq 2$

On Combining, we get $\left(\sin x + \frac{1}{\sin x}\right) + \left(\tan y + \frac{1}{\tan y}\right) > 4$

which is not possible according to (i)

Therefore $\sin x + \frac{1}{\sin x} = 2$ and $\tan y + \frac{1}{\tan y} = 2$

$\Rightarrow \sin x = 1$ and $\tan y = 1$

$\Rightarrow x = \pi/2$ and $\frac{2\tan y/2}{1-\tan^2 y/2} = 1$

$\Rightarrow 2\tan y/2 = 1 - \tan^2 y/2$

$\Rightarrow \tan^2 y/2 + 2\tan y/2 - 1 = 0$

$\Rightarrow \tan y/2$ is root of $\alpha^2 + 2\alpha - 1 = 0$ B

**5. (C)** $f(x) = \begin{cases} -1-x & -x \leq -1 \\ x+1 & -1 < x \leq 0 \\ 1-x & 0 < x \leq 1 \\ x-1 & x > 1 \end{cases}$

At $x = -1$

RHL $= \dfrac{(-1+h+1)-0}{h} = 1$

LHL $= \dfrac{-1-(-1-h)-0}{-h} = -1$

$\Rightarrow$ at $x = 0, x = 1$

$LHL \neq RHL$

**6. (C)** $\displaystyle \log_{y\to\infty} \frac{\int_1^y \left[\tan^{-1} x\right] dx}{\int_1^y \left[1+\frac{1}{x}\right] dx}$

$\displaystyle \int_1^y \left[\tan^{-1} x\right] dx = y - \tan 1$ and $\displaystyle \int_1^y \left[1+\frac{1}{x}\right] dx = y-1$

$\displaystyle \Rightarrow \lim_{y\to\infty} \frac{y-\tan 1}{y-1} = 1$

**7. (ABCD)** $f'(x) = \sqrt{1-x^4} > 0\ \forall\ x \in [-1,1]$ increasing

Now $f(x)+f(-x)=\left[\int_0^x \sqrt{1-x^4}+\int_0^{-x}\sqrt{1-x^4}\right].dx=\left[\int_0^x \sqrt{1-x^4}dx+\int_0^t \sqrt{1-t^4}(-dt)\right]$

Put $-x=t=0$

$=o \;\Rightarrow\; f(x)$ is odd

Again $f''(x)=\dfrac{-4x^3}{2\sqrt{1-x^4}}$ which vanishes at x = 0 and changes *sig n*

$\Rightarrow (0,0)$ is point of inflection

**8. (CD)** NA

**9. (A)** N.A.

**10. (CD)** $I=\int_0^\infty \dfrac{sin^4 x\,.\,dx}{x^2}$

By using by parts we get

$I=sin^4 x\left(\dfrac{-1}{x}\right)\Big|_0^\infty+\int_0^\infty \dfrac{4\,sin^3 x.cos\,x}{x}.dx$

$I=0+\int \dfrac{4\,sin^3 x\,.\,cos\,x}{x}.dx$

$I=\int_0^\infty \dfrac{(3\,sin\,x-sin\,3x)cos\,x}{x}$

$I=\dfrac{3}{2}\int_0^\infty \dfrac{sin\,2x}{x}.dx-\dfrac{1}{2}\int \dfrac{sin\,4x+sin\,2x}{x}.dx$

$=\dfrac{3}{2}J-\dfrac{1}{2}(J+J).dx$

$=J/2$

As $\int_0^\infty \dfrac{sin\,2x}{x}.dx=J$ and $\int_0^\infty \dfrac{sin\,4x}{x}.dx=J$

By putting $2x=t$ and $4x=t$

**11. (A)** Algebraically $(5+i)^4=476+480i$ ……………… (I)

and by polar representation of $5+i$

$(5+i)^4=676(cos\,4\theta+i\,sin\,4\theta)$ ……………… (II)

Equating the arguments of $(5+i)^4$ from (I) and (II) we get

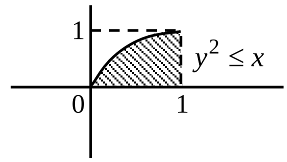
$tan^{-1}\dfrac{480}{476}=4\theta=4\,tan^{-1}\dfrac{1}{5}$

As $\dfrac{480}{476}$ is nearly equal to 1

$\Rightarrow 4tan^{-1}\frac{1}{5} = \frac{\pi}{4}$

$\Rightarrow tan^{-1}\frac{1}{5} = \frac{\pi}{16}$

**12. (A)** Area of region = $\int_0^1 \sqrt{x}.\,dx = \frac{2}{3}$




$\Rightarrow$ Required probability $= \frac{2/3}{1} = \frac{2}{3}$

$\Rightarrow$ Statement I is correct and Statements II is correct Explanation.

**13. (A)** LH rule is applicable for statement I and after applying it, the limit reduces to statement II

**14. (A)** $sin\frac{A}{2} \le \frac{a}{b+c} \Rightarrow sin\frac{A}{2} \le \frac{sin\,A}{sin\,B + sin\,C}$

$$\Rightarrow sin\frac{A}{2} \le \frac{2\,sin\frac{A}{2}.cos\frac{A}{2}}{2\,sin\frac{B+C}{2}.cos\frac{B-C}{2}}$$

$$= cos\left(\frac{B-C}{2}\right) \le 1$$

Statement II is true

$cos\frac{B-C}{2} \ge 0$ is always true in a triangle

As $\frac{B-C}{2} \in \left(\frac{-\pi}{2}.\frac{\pi}{2}\right)$

**15. (B)** Put $x = 0$ in the first equation we get $t = 0, t = 1, t = -1$

for $t = 0, y = 1 - 0^4 = 1$

for $t = 1, y = 1 - 1^4 = 0$

for $t = -1, y = 1 - (-1)^4 = 0$

$\Rightarrow$ Curve cuts y-axis at two points $(0,0)$ and $(0,1)$.

**16. (B)** If we replace t $by -t$, y remains same but $x$ becomes – x

$\Rightarrow$ function is symmetric about y-axis

**17. (B)** Area of the loop $= \left| 2\int_0^1 x\,dy \right| = \left| 2\int_0^1 x\frac{dy}{dt}dt \right|$

$$= \left| 2\int_0^1 (t - t^3)(-4t^3)dt \right| = \frac{16}{35}$$

**18. (D)** $\alpha + \beta + \gamma = -a$

$\alpha\beta + \beta\gamma + \gamma\alpha = b$

$\alpha\,\beta\,\gamma = -c$

$\frac{\alpha+\beta+\gamma}{3} \ge (\alpha\,\beta\,\gamma)^{\frac{1}{3}} = (-C)^{\frac{1}{3}} = +\frac{1}{4}$

$\Rightarrow \quad \alpha+\beta+\gamma \geq \frac{3}{4}$

$\Rightarrow \; min(\alpha+\beta+\gamma)=\frac{3}{4}$

**19. (C)** If $\alpha=-1$ then $\alpha+\beta+\gamma=1$

$\Rightarrow \qquad \alpha+\frac{\beta}{2}+\frac{\beta}{2}+\frac{\gamma}{3}+\frac{\gamma}{3}+\frac{\gamma}{3}=1$

$\Rightarrow \qquad max\left(\alpha \cdot \frac{\beta^2}{4} \cdot \frac{\gamma^3}{27}\right)$

Will be attained if $\alpha=\frac{\beta}{2}=\frac{\gamma}{3}=\frac{1}{6}$

or $\qquad \alpha=\frac{1}{6}, \beta=\frac{1}{3}, \gamma=\frac{1}{2}$

$\Rightarrow \qquad max\left(\alpha\,\beta^2\,\gamma^3\right)=\frac{1}{6}\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)^3$

$=\frac{1}{432}$

**20. (A)** $(\alpha+\beta+\gamma)^3 \geq 27\,\alpha, \beta, \gamma$ is essentially true $(\because AM \geq GM)$

$\Rightarrow \qquad (\alpha\,\beta\,\gamma)^3-27\alpha\,\beta\,\gamma \leq 0$ is positive only when

$\Rightarrow \qquad (\alpha+\beta+\gamma)^3=27\alpha\,\beta\,\gamma$

Whence $\alpha=\beta=\gamma=\frac{1}{4}$

$\Rightarrow \qquad a+b=-(\alpha+\beta+\gamma)+\alpha\beta+\beta\gamma+\gamma\alpha$

$=-\frac{3}{4}+\frac{3}{16}=-\frac{9}{16}$

**21.(D)** $a_1=0, a_{n-1}=a_n^2-i$

$a_2=a_1^2-i=-i$

$a_3=a_2^3-i=(-i)^2-i=-1-i$

$a_4=(-1-i)^2-i=1+i^2+2i-i=i$

$a_5=(i)^2-i=-1-i$ (Same as $a_3$)

$a_6=(-1-i)^2-i$ (Same as $a_4$)

$=i$

Thus $a_{2n}=i$ if $n>1, a_{2n+1}=-1-i$ if $n>1$

**22. (C)**

**23. (D)** $a_{2008}=i, a_{2009}=-1-i$

$\Rightarrow$ Points are $(0,1)$ and $(-1,-1)$.

The distance between them is $\sqrt{5}$.